

The Gauge-Bethe Correspondence and Geometric Representation Theory

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ABSTRACT

The Gauge/Bethe correspondence of Nekrasov and Shatashvili relates the spectrum of integrable spin chains to the ground states of supersymmetric gauge theories. Up to now, this correspondence has been an observation; the underlying reason for its existence remaining elusive. We argue here that geometrical representation theory is the mathematical foundation of the Gauge/Bethe correspondence, and it provides a framework to study families of gauge theories in a unified way.

1 Introduction

In [1, 2] the *Gauge/Bethe correspondence* was introduced, relating integrable spin chains to $2d$ supersymmetric gauge theories. The duality was further extended to four dimensional gauge theories in [3]. In [4], it was used to relate the ground states of different quiver gauge theories. In the simplest case of a $\mathfrak{su}(2)$ $\text{xxx}_{1/2}$ spin chain, the statement of this correspondence can be summarized as follows: the spectrum of the N magnon sector of the integrable system corresponds to the supersymmetric ground states of a matching $\mathcal{N} = (2, 2)$ $\text{U}(N)$ gauge theory in two dimensions. For an $\text{xxx}_{1/2}$ spin chain of length L , the low energy limit of the corresponding gauge theory is the non-linear sigma model (NLSM) with target space the cotangent bundle of the Grassmannian $T^*\text{Gr}(N, L)$ [2].

Since the total spin is a symmetry of the spin chain, one usually decomposes the Hilbert space of the spin chain into N *magnon* sectors, *i.e.* sectors with N down spins and total spin $L/2 - N$. To each such sector with N magnons corresponds a $\text{U}(N)$ gauge theory via the Gauge/Bethe construction. To describe the full Hilbert space and the full spectrum of the spin chain, of course all N magnon sectors with $N = 0, \dots, L$ need to be considered. In gauge theory, on the other hand, one usually studies each $\text{U}(N)$ by itself, and not different values of N together. There is, however, evidence that doing so would make sense. In the case of the $\text{xxx}_{1/2}$ spin chain, there is an obvious equivalence between taking the state with all spins up as the reference state and considering the sector with N down spins, and taking the state with all spins down as reference state and considering the sector with $L - N$ up spins instead. This equivalence is on the gauge theory side reflected in the *Grassmannian duality* relating $\text{Gr}(N, L)$ and $\text{Gr}(L - N, L)$. This is the simplest instance of a relation between gauge theories with different N .

We would like to argue that taking the Gauge/Bethe correspondence seriously, one should look for a framework in which it makes sense to consider gauge theories with different N together. Moreover, we learn from the integrable model side of the correspondence that this set of gauge theories must carry a $\mathfrak{su}(2)$ symmetry, a fact that only becomes obvious in such a unified framework. The integrable structure of the spin chain remains hidden as long as the gauge theories are considered separately, but will be manifest in a framework that unifies them in a meaningful way. In fact, such a mathematical framework exists and goes under the name of *geometric representation theory*. It goes back to Victor Ginzburg and is explained in detail in his book [5]. Up to now, the Gauge/Bethe correspondence has been an observation; the underlying reason for its existence remaining elusive. Geometrical representation theory can be understood as the mathematical foundation or underlying reason for the Gauge/Bethe correspondence.

We will be explicitly treating the $\mathfrak{su}(2)$ Heisenberg spin chain, which is the simplest case. However, our arguments extend directly to $\mathfrak{su}(n)$ spin chains for any $n \in \mathbb{N}$. The corresponding sigma models in this case have as target space the cotangent bundle of a *flag variety*. While the Gauge/Bethe correspondence is applicable to spin chains with *any* Lie group or even supergroup symmetry, the mathematical construction of the geometric representation theory for the general case is beyond the scope of this note.

The plan of this note is as follows. In Sec. 2, we briefly summarize the Gauge/Bethe correspondence, leaving the details to the original literature. In Sec. 3, geometric representation theory is introduced and applied to the Gauge/Bethe correspondence. In Sec. 4, the implications of the above and further directions of study are explored.

2 The Gauge/Bethe correspondence

The Gauge/Bethe correspondence, as detailed in [1, 2], relates two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories to quantum integrable systems. The supersymmetric vacua of the gauge theories form a representation of the *chiral ring*, which is a distinguished class of operators which are annihilated by one chirality of the supercharges Q . The *commuting Hamiltonians* of the quantum integrable system are identified with the generators of the chiral ring. The space of states of the quantum integrable system, *i.e.* the spectrum of the commuting Hamiltonians, is thus mapped to the supersymmetric vacua of the gauge theory. Expressed differently, the *effective twisted superpotential* of the gauge theory in the Coulomb branch is identified with the *Yang–Yang counting function* which serves as a potential for the Bethe equations, whose solutions are the spectrum of the integrable system. This is true sector by sector. The dictionary between a general spin chain and the corresponding gauge theory was detailed in [4]. Here, we will confine ourselves to a very simple case, namely the one of a $\mathfrak{su}(n)$ (mostly $n = 2$) Heisenberg spin chain of length L with periodic boundary conditions and no inhomogeneities, where each position carries the fundamental representation of $\mathfrak{su}(n)$. Each position in the chain admits one of $n - 1$ different particle species. If we focus on the sector containing N_a particles of species a , this model results in an A_{n-1} quiver gauge theory with the following properties: each node $a = 1, \dots, r$ carries a $U(N_a)$, $N_a \in \{1, \dots, L\}$ gauge group. The bifundamental matter fields between neighboring nodes have twisted mass $m_B = -\iota/2$, the adjoint matter fields occurring at each node have twisted mass $m_\Phi = \iota$. One of the $n - 1$ nodes is connected to a $U(L)$ flavor group, and the fundamental and anti-fundamental matter field have twisted mass $m_Q = -\iota/2$.

The simplest case, namely that of the $\mathfrak{su}(2)$ xxx spin chain of length L , corresponds to the $\mathcal{N} = (2, 2)$ theory with gauge group $U(N)$, flavor group $U(L)$ and the following matter content:

- an adjoint field Φ with twisted mass ι ,
- L fundamentals and anti-fundamentals Q_n, \bar{Q}_n with twisted mass $-\iota/2$.

We are interested in the *Coulomb branch* of the theory: we therefore consider the low energy effective theory obtained for slowly varying σ fields¹, after integrating out the massive matter fields. The resulting vacuum manifold is the cotangent bundle to the Grassmannian $T^*\text{Gr}(N, L)$:

$$\text{Gr}(N, L) = \{ W \subset \mathbb{C}^L \mid \dim W = N \} , \quad (2.1)$$

$$T^*\text{Gr}(N, L) = \{ (X, W), W \in \text{Gr}(N, L), X \in \text{End}(\mathbb{C}^L) \mid X(\mathbb{C}^L) \subset W, X(W) = 0 \} . \quad (2.2)$$

¹The complex scalar σ is the lowest component of the super-field strength, which is a twisted chiral multiplet.

Via the effective twisted superpotential

$$\begin{aligned} \tilde{W}_{\text{eff}}^N(\sigma) = & \frac{L}{2\pi} \sum_{i=1}^N \left[\left(\sigma_i + \frac{\imath}{2} \right) \left(\log(\sigma_i + \frac{\imath}{2}) - 1 \right) - \left(\sigma_i - \frac{\imath}{2} \right) \left(\log(-\sigma_i + \frac{\imath}{2}) - 1 \right) \right] \\ & + \frac{1}{2\pi} \sum_{\substack{i,j \\ i \neq j}}^N (\sigma_i - \sigma_j - \imath) \left(\log(\sigma_i - \sigma_j - \imath) - 1 \right) - \imath \tau \sum_{i=1}^N \sigma_i, \end{aligned} \quad (2.3)$$

we can obtain the vacua of the theory which are the solutions of the equation [6]

$$\exp \left[2\pi \frac{\partial \tilde{W}_{\text{eff}}(\sigma)}{\partial \sigma_i} \right] = 1. \quad (2.4)$$

The minima satisfy the equation

$$\left(\frac{\sigma_i + \imath/2}{\sigma_i - \imath/2} \right)^L = \prod_{\substack{j=1 \\ j \neq i}}^N \frac{\sigma_i - \sigma_j + \imath}{\sigma_i - \sigma_j - \imath} \quad i = 1, 2, \dots, N. \quad (2.5)$$

Because of the presence of twisted masses, the corresponding ground states generate the equivariant cohomology of $H_{\text{SU}(L) \times \text{U}(1)}^*[T^*\text{Gr}(N, L)]$ which in this case (*i.e.* no inhomogeneities in the spin chain) is isomorphic to the standard cohomology.

The other side of the correspondence is the $\text{xxx}_{1/2}$ spin chain, *i.e.* a system of L spins on a circle, each of which carries the fundamental representation $V = \mathbb{C}^2$ of $\text{su}(2)$. The total Hilbert space is given by the product

$$\mathcal{H} = \bigotimes_{n=1}^L V \simeq (\mathbb{C}^2)^{\otimes L}, \quad (2.6)$$

and there is a natural action of $\text{su}(2)$ on \mathcal{H} , generated by

$$E = \sum_{m=1}^L \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{m-1} \otimes e \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad (2.7)$$

$$F = \sum_{m=1}^L \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{m-1} \otimes f \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad (2.8)$$

$$K = \sum_{m=1}^L \underbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}_{m-1} \otimes k \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad (2.9)$$

where e, f, k generate the fundamental representation of $\text{su}(2)$:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.10)$$

Physically, E flips a spin up, F flips a spin down and K is twice the total spin. Thanks to

the $\mathfrak{su}(2)$ invariance of the Hamiltonian, the Hilbert space decomposes into the direct sum of subspaces with fixed number of spins up:

$$\mathcal{H} = \bigoplus_{N=0}^L V_{L-2N}, \quad (2.11)$$

$$V_{L-2N} = \{ \psi \in \mathcal{H} \mid K \psi = (L - 2N) \psi \}. \quad (2.12)$$

The vectors $\psi \in V_{L-2N}$ contain N excitations, or magnons. A basis of the weight space V_{L-2N} is obtained as the set of states depending on N rapidities λ_i that satisfy the Bethe Ansatz equation²:

$$\left(\frac{\lambda_i + \iota/2}{\lambda_i - \iota/2} \right)^L = \prod_{\substack{j=1 \\ j \neq i}}^N \frac{\lambda_i - \lambda_j + \iota}{\lambda_i - \lambda_j - \iota}, \quad i = 1, 2, \dots, N. \quad (2.13)$$

This is precisely the same equation that we had found above for the ground states of the non-linear sigma model once we identify the field σ_i with the rapidity λ_i . This observation is the essence of the Gauge/Bethe correspondence. Note that the solutions to the Bethe Ansatz equation, which we find on both sides of the correspondence, parametrize the *eigenvectors* of the $\text{xxx}_{1/2}$ chain and not the eigenvalues of the Hamiltonian. We would like to stress that the properties of an integrable spin chain are determined by its symmetry group (including the information about its boundary conditions and the representation of the symmetry group each position is carrying). Its Hamiltonian is a consequence of the Yang–Baxter equation which is determined by the symmetries of the chain.

Given the Gauge/Bethe correspondence we can use techniques from representation theory (the xxx side) to answer gauge theory questions. The completeness of the Bethe equation for example implies that the generating function for the number of solutions in each sector V_{L-2N} is given by the character of q^K . The latter decomposes into the product over the L fundamental representations:

$$Z(q) = \text{Tr}_{\mathcal{H}} q^K = \prod_{n=1}^L \text{Tr}_V q^n = q^L \prod_{n=1}^L (1 + q^{-n}) = \sum_{N=0}^L \binom{L}{N} q^{L-2N}. \quad (2.14)$$

Via the correspondence, this is also the generating function for the dimensions of the cohomologies of the cotangent bundles to the Grassmanians. Hence

$$\dim H^*[T^*\text{Gr}(N, L)] = \binom{L}{N}. \quad (2.15)$$

²In the case of twisted boundary conditions, the Bethe Ansatz equation admits $\binom{L}{N}$ solutions. For periodic b.c. there are only $\binom{L}{N} - \binom{L}{N-1}$ solutions, corresponding to the highest weight states of $\mathfrak{su}(2)$ in V_{L-2N} ; these have to be supplemented by the descendants of the states in V_{L-2N+2} . For details see e.g. [7]. This is consistent with the fact that the geometric interpretation of the gauge theory requires a finite FI term.

3 Geometric representation theory for $\mathfrak{su}(2)$

We now introduce the main ideas of geometric representation theory. Precise details can be found in the book [5].

A representation of the algebra $\mathfrak{su}(2)$ consists in a vector space V and an action of three operators e, f, k satisfying the relations

$$[e, f] = k, \quad [k, e] = 2e, \quad [k, f] = -2f. \quad (3.1)$$

Given the tensor product of L copies of the fundamental representation V , there is a natural inclusion of the $(L + 1)$ -dimensional irreducible representation, $V(L) \hookrightarrow V^{\otimes L}$. Geometric representation theory provides a construction for $V^{\otimes L}$ where the vector space is the direct sum of the homologies of the cotangent bundles to all the Grassmannians $T^*\mathrm{Gr}(N, L)$ for fixed L :

$$V^{\otimes L} \simeq \bigoplus_{N=0}^L H_*[T^*\mathrm{Gr}(N, L), \mathbb{C}] = H_*[T^*\mathrm{Gr}(L), \mathbb{C}], \quad (3.2)$$

where $\mathrm{Gr}(L)$ is the disjoint union of all the Grassmannians $\mathrm{Gr}(N, L)$ for fixed L :

$$\mathrm{Gr}(L) = \bigsqcup_{N=0}^L \mathrm{Gr}(N, L) \quad (3.3)$$

(we set $\mathrm{Gr}(0, L) = \mathrm{Gr}(L, L) = \emptyset$). Each of the terms $H_*[T^*\mathrm{Gr}(N, L), \mathbb{C}]$ is identified with the $(L - 2N)$ weight space, which has dimension $\binom{L}{N}$:

$$H_*[T^*\mathrm{Gr}(L)] \simeq V^{\otimes L} = \bigoplus_{N=0}^L V_{L-2N} \simeq \bigoplus_{N=0}^L H_*[T^*\mathrm{Gr}(N, L)]. \quad (3.4)$$

The key point of the construction is the definition of the operators e and f which act between the homologies,

$$e, f : H_*[T^*\mathrm{Gr}(L)] \rightarrow H_*[T^*\mathrm{Gr}(L)]. \quad (3.5)$$

In particular, we need f to act between two components of $\mathrm{Gr}(L)$, raising N by 1:

$$f : H_*[T^*\mathrm{Gr}(N, L)] \rightarrow H_*[T^*\mathrm{Gr}(N + 1, L)]. \quad (3.6)$$

This is possible by introducing a *correspondence* (in the mathematical sense)

$$\begin{array}{ccc} & Z \subset T^*\mathrm{Gr}(N, L) \times T^*\mathrm{Gr}(N + 1, L) & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ T^*\mathrm{Gr}(N, L) & & T^*\mathrm{Gr}(N + 1, L) \end{array} \quad (3.7)$$

where Z is the diagonal part of the cotangent bundle of the product of two Grassmannians:

$$Z = \{ (X, U_N, U_{N+1}) \mid U_i \in \text{Gr}(N_i, L), X \in \text{End}(\mathbb{C}^L), U \subset U', X(\mathbb{C}^L) \subset U, X(U') = 0 \} . \quad (3.8)$$

We can now define the *Hecke operator* f by first acting with the pullback π_1^* , then intersecting with the fundamental class $[Z]$ and finally acting with the pushforward π_{2*} :

$$f: H_*[T^*\text{Gr}(N, L)] \rightarrow H_*[T^*\text{Gr}(N+1, L)] \quad (3.9)$$

$$x \mapsto f(x) = \pi_{2*}([Z] \cap \pi_1^*(x)) . \quad (3.10)$$

The operator e is defined in a similar way,

$$e: H_*[T^*\text{Gr}(N+1, L)] \rightarrow H_*[T^*\text{Gr}(N, L)] \quad (3.11)$$

$$x \mapsto e(x) = (-1)^L \pi_{1*}([Z] \cap \pi_2^*(x)) . \quad (3.12)$$

One can prove that the commutation relation $[e, f] = k$ is satisfied if k is the operator that multiplies each component $H_*(T^*\text{Gr}(N, L))$ by $(L - 2N)$:

$$k = \bigoplus_{N=0}^L (L - 2N) \text{Id}_{H_*[T^*\text{Gr}(N, L)]} , \quad (3.13)$$

which shows that the $L - 2N$ weight space is precisely the homology of $T^*\text{Gr}(N, L)$:

$$H_*[T^*\text{Gr}(N, L)] = \{ x \in H_*[T^*\text{Gr}(L)] \simeq V^{\otimes L} \mid kx = (L - 2N)x \} \simeq V_{L-2N} . \quad (3.14)$$

In particular one finds that the homology $H_{top}[T^*\text{Gr}(L)]$ (the span of the fundamental classes of all the Grassmannians of \mathbb{C}^L , $[T^*\text{Gr}(N, L)]$) is stable under the action of e and f and hence provides the highest weight representation of dimension $L + 1$,

$$H_{top}[T^*\text{Gr}(L)] = \bigoplus_{N=0}^L H_{top}[T^*\text{Gr}(N, L)] = \bigoplus_{N=0}^L [T^*\text{Gr}(N, L)] \simeq V(L) . \quad (3.15)$$

Let us try to interpret this construction in terms of physical systems. The direct sum of the homologies of the cotangent bundles over the Grassmannians of \mathbb{C}^L corresponds to the ground states of the non-linear sigma models on all the $T^*\text{Gr}(N, L)$ for $N = 0, 1, \dots, L$. Via geometric representation theory, this space can be given the structure of the $V^{\otimes L}$ representation of $\text{su}(2)$ (Eq. (3.2)). The Hilbert space of the $\text{xxx}_{1/2}$ spin chain (Eq. (2.6)) has the same structure. We can identify the two spaces. The homology of the Grassmannian $H_*[T^*\text{Gr}(N, L)]$ (Eq. (2.5)) is the $(L - 2N)$ weight space V_{L-2N} (Eq. (3.14)), which is spanned by the spectrum of the $\text{xxx}_{1/2}$ chain in the N magnon sector (Eq. (2.13)). It follows that there is a one-to-one correspondence between the minima of the twisted superpotential of the NLSM and the solutions to the Bethe Ansatz (gauge/Bethe correspondence). In Table 1 we provide a minimal dictionary between the gauge Bethe correspondence and geometric representation theory.

The construction can be generalized to A_{n-1} algebras which are represented by

physics	mathematics
spectrum of $\text{xxx}_{1/2}$ spin chain	$\text{su}(2)$ representation $V^{\otimes L} \simeq H_*[T^*\text{Gr}(L)]$
ground states of the NLSM on $T^*\text{Gr}(N, L)$	cohomology $H^*[T^*\text{Gr}(N, L)]$
spectrum for the N magnon sector	weight space $V_{L-2N} \simeq H_*[T^*\text{Gr}(N, L)]$
ground states of $\text{xxx}_{1/2}$	hw representation $V(L) \simeq H_{top}[T^*\text{Gr}(L)]$
gauge/Bethe correspondence	geometric representation of su_2

Table 1: Dictionary

homologies of cotangent bundles to flag spaces $H_*[T^*\text{Fl}_n\mathbb{C}^L]$:

$$\text{Fl}_\mu\mathbb{C}^L = \{ 0 = U_0 \subset U_1 \subset \cdots \subset U_n = \mathbb{C}^L, \mu \in \mathbb{N}^n \mid \dim U_i - \dim U_{i-1} = \mu_i \} , \quad (3.16)$$

$$T^*\text{Fl}_\mu\mathbb{C}^L = \{ (X, U_\bullet) \mid U_\bullet \in \text{Fl}_\mu\mathbb{C}^L, X(U_i) \subset U_{i-1} \} , \quad (3.17)$$

$$\text{Fl}_n\mathbb{C}^L = \bigsqcup_{\mu_1 + \mu_2 + \cdots + \mu_n = L} \text{Fl}_\mu\mathbb{C}^L . \quad (3.18)$$

These are the vacuum manifolds for the quiver gauge theories identified in Sec. 2.

4 Further directions

This note is a starting point for a wealth of generalizations and further developments. Our statement readily generalizes to $\text{su}(n)$, but the scope of the Gauge/Bethe correspondence is much wider and includes spin chains with different symmetry groups, inhomogeneities, etc. While a number of mathematical difficulties need to be conquered in order to get there, we expect the formalism of geometric representation theory to be powerful enough to encompass all of the above. We would like to conclude this note by pointing out some directions for future research, based on the identification between the Gauge/Bethe correspondence and geometric representation theory.

1. The central object of the Gauge/Bethe correspondence is the Bethe Ansatz equation which is a consequence of the Yang–Baxter relations. These are in turn related to representations of the *Yangian algebra* for which the spin chain admits a natural action [7]. This points to a representation of the Yangian in terms of Grassmannians, generalizing the construction of Sec. 3, which would further elucidate the role of integrability.
2. The Gauge/Bethe correspondence associates products of general representations to *equivariant* cohomologies. This calls for a generalization of the construction in Sec. 3 to the equivariant case.
3. Bethe Ansatz equations of the form (2.13) can be written for quantum integrable systems with any Lie group, or supergroup symmetry. To capture the general case, geometric representations of all these groups are necessary.
4. The space of solutions of $\text{xxx}_{1/2}$ admits a natural grading in terms of *rigged partitions* [8]. The corresponding fermionic formula reproduces the generating function for the Poincaré polynomials of the Grassmannians $\text{Gr}(L)$.

Geometric representation theory is moreover closely related to Nakajima's work on quiver varieties [9, 10], which opens up further interesting avenues of study.

Perhaps the most important message is that it makes sense to consider a unified framework in which a set of gauge theories (e.g. all the $U(N)$ gauge theories with L flavors and $N = 0, 1, \dots, L$) are studied together. Only in this setting, the $su(2)$ algebra (and in general the integrable structure) will become manifest. In a string theory interpretation, the Hecke operators e and f defined in Sec. 3 could be naturally understood as brane creation and annihilation operators. But this goes beyond the scope of this note.

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